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Degree of Approximation for Functions in the Generalised Zygmund Class

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ABSTRACT

Fourier partial sums may converge slowly or exhibit oscillatory behaviour for functions with limited smoothness, which makes summability methods important in quantitative approximation. This study derives an error estimate for 2π-periodic functions in a generalised Zygmund class when their Fourier series is processed by the Taylor–Cesàro (Tn, C2) product mean. The analysis formulates the associated product kernel, establishes separate bounds for small and large arguments, and applies the generalised Minkowski inequality together with the monotonicity of t^α ω(t). The resulting estimate extends the Cesàro-based theorem of Kim by replacing a single averaging operator with a regular product procedure that permits Taylor weighting and second-order Cesàro smoothing. The asymptotic order is not claimed to be strictly faster in every case; the main gain is a broader summability framework and greater flexibility in damping oscillatory partial sums. For ω(t) = t^m (m > 1), explicit bounded, logarithmic, and algebraic rates follow according to the value of α + m. The result clarifies the operator-theoretic role of product summability in Fourier approximation and supports extensions to weighted, deferred, and multidimensional settings.

1. INTRODUCTION

Quantifying how well trigonometric polynomials reproduce a periodic function is a central problem in Fourier approximation. Classical studies relate the approximation error to smoothness conditions such as Lipschitz and Zygmund regularity, while summability procedures replace unstable partial sums by averaged transforms that are better suited to convergence analysis [1-3].

Generalised Zygmund classes provide a flexible description of second-order smoothness through a modulus function and an L^r metric. They therefore accommodate a wider range of local behaviour than a single Hölder exponent. Matrix, Hausdorff, Euler, and related operators have been used to obtain approximation estimates in these spaces [4-5].

Recent work has increasingly examined product and deferred summability schemes because a second averaging stage can offer additional control over oscillatory Fourier partial sums. Representative developments include Euler–Hausdorff, deferred Euler–Nörlund, delayed-mean, and related product operators [2, 4, 6,7]. These studies motivate a careful assessment of what is genuinely gained by replacing a classical Cesàro mean with a product construction.

Beyond classical analysis, summability and approximation techniques underpin the signal-processing pipelines on which modern algorithmic and data-driven systems rely; the responsible governance of such systems and their implications for the future of work have accordingly become pressing interdisciplinary concerns [8].

Kim [6] established a Cesàro estimate for functions in a generalised Zygmund class. Taylor–Cesàro product methods have also been considered for Lipschitz-type classes, but the corresponding generalized-Zygmund estimate and its relation to Kim's bound require a unified presentation [9-11]. The objective of this paper is therefore to derive the degree of approximation produced by the Taylor–Cesàro. (T_n, C_2) product mean, state the assumptions precisely, and distinguish broader operator generality from any claim of a universally faster asymptotic rate [12-14].

2. PRELIMINARIES AND NOTATION

2.1 Fourier series and approximation error

Let f be a 2π-periodic function that is Lebesgue integrable on $(-\pi,\ \pi)$. Its Fourier series is written as

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad (1)$$

With partial sum $S_n(x)$.

The degree of approximation $E_n(f)$ of a function $f : R \to R$ by a trigonometric polynomial t_n of degree n is defined by

$$E_n(f) = \|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} \text{ [21]} \qquad (2)$$

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^\alpha)$, for $0 < \alpha \le 1$.

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose nth partial sum is given by $S_n = \sum_{k=0}^{\infty} u_k$.

2.2 Cesàro and Taylor–Cesàro summability

Cesaro means $(C, 1)$ of sequence $\{S_n\}$ is given by $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$. If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesaro means $(C, 1)$ to S. It is denoted by $\sigma_n \to S((C, 1))$, as $n \to \infty$.

Let $\sum u_n$ be a given infinite series with sequence of its nth partial sum $\{S_n\}$. The $(C, 2)$ transform is defined as the nth partial sum of $(C, 2)$ summability and is given by

$$\sigma_n = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)S_k \to S \text{ as } n \to \infty \qquad (3)$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite numbers by $(C, 2)$ method.

A given sequence $\{S_n\}$ is said to be Taylor summable, if $(T_n) = \sum_{k=0}^{n} u_{n,k}\, S_k \to S$ as $n \to \infty$, then the $(C, 2)$ transform of Taylor means defines the (T_n, C_2) transform of the partial sum $\{S_n\}$ of the series (2.1).

Thus, if $(T_n, C_2) = \sum_{k=0}^{n} u_{n,n-k}, \sigma_{n-k} \to S$ as $n \to \infty$

then $\sum_{n=0}^{\infty} u_n$ is said to be T_n, C_2 summable to S.

REMARK: - We shall use the following notations:

(i) $\varnothing(t) = f(x+t) - f(x-t) - 2f(x)$

(ii) $D(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{u_{n,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}$

2.3 Generalised Zygmund class and product kernel

Let L_r norm of a function f is defined

$$\|f\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^r dx\right)^{\frac{1}{r}}, & 1 \le r < \infty \\ ess \sup_{0<x\le 2\pi}|f(x)|, & r = \infty \end{cases}$$

The Zygmund modulus of continuity of f is defined by

$$\omega(f, h) = \sup_{0\le h, x\in R}|f(x+t) + f(x-t)|$$

Let $C_{2\pi}$ be the Banach space of all 2π – periodic functions (continuous) defined on $[0,2\pi]$ under the supremum norm. For $0 < \alpha \le 1$, the Zygmund class $Z_{uv}(\alpha, r)$ is defined as

$$Z_{uv}(\alpha, r) = \{f \in C_{2\pi} : |f(x+t) + f(x-t) - 2f(x)| = O(|t|))\}$$

is also a Banach space with the norm$\|.\|_{(\alpha)}$, given by

$$\|f\|_{(\alpha)} = \sup_{0\le x\le 2\pi}|f(x)| + \sup_{x,t\ne 0}\frac{|f(x+t) + f(x-t)|}{|t|^\alpha}$$

For $f \in L_r[0,2\pi], r \ge 1$, the integral Zygmund modulus of continuity is given by

$$w_r(f,h) = \sup_{0<t\le h}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(x+t)+\ f(x-t)|^r dx\right\}^{\frac{1}{r}} \qquad (4)$$

For $f \in C_{2\pi}$ and $r = \infty$.

We introduce a generalised Zygmund class.

$$Z_{uv^{\omega}}(\alpha,r) = \{f \in C_{2\pi}: |f(x+t)+\ f(x-t)-2f(x)| = O(|t|^{\alpha})\omega(|t|))\}$$

Where $\alpha \ge 0,\ r \ge 1$ and w is a continuous, nonnegative and nondecreasing function. If we take $\alpha = 1$, $\omega =$ constant and $r \to \infty$, then $Z_{uv^{\omega}}(\alpha,r)$ class reduces to the $Z_{uv}(\alpha,r)$.

3. CESÀRO REFERENCE RESULT

Kim [6] examined approximation in a generalised Zygmund class by first-order Cesàro means. In the notation used here, the reference result may be stated as follows.

Theorem 1 (Kim [6]). Let f be a continuous 2π-periodic function in the relevant generalised Zygmund class. Then its first-order Cesàro means satisfy.

Let f be a 2π periodic continuous function belonging to the Generalised Zygmund class $Z_{yg^{\omega}}(\alpha,\gamma)$. Then the degree of approximation of f by Cesaro means of its Fourier series is given by

$$\|\sigma_n - f\|_{\gamma} = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \qquad (5)$$

4. MAIN RESULT AND ANALYTICAL COMPARISON

The Taylor–Cesàro construction replaces a single averaging stage by a regular Taylor transform of second-order Cesàro means. Under the coefficient conditions stated in Section 2.2, the following result extends the Cesàro estimate to this product operator.

Theorem 2. Let f be a continuous 2π-periodic function belonging to the generalised Zygmund class defined in Section 2.3. Then the degree of approximation generated by the Taylor–Cesàro product mean satisfies

$$\|\mathbf{t_n}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|_{\mathbf{r}} = \mathbf{O}\left(\frac{\mathbf{1}}{\mathbf{n+1}}\sum_{\mathbf{j=1}}^{\mathbf{n+1}}\left(\frac{\mathbf{1}}{\mathbf{j}}\right)^{\boldsymbol{\alpha}}\boldsymbol{\omega}\left(\frac{\mathbf{1}}{\mathbf{j}}\right)\right) \qquad (6)$$

4.1 Comparison with the classical Cesàro estimate

Theorem 2 has the same displayed asymptotic order as the benchmark in Theorem 1 under the stated modulus assumptions. Consequently, the present result should not be interpreted as proving a strictly smaller error for every admissible function. Its contribution is operator-theoretic to the Taylor weights and the second-order Cesàro stage create a broader regular averaging framework in which the classical type of estimate remains valid.

Table 1. Analytical comparison of the Cesàro and Taylor–Cesàro constructions.

Aspect	Cesàro (C,1)	Taylor–Cesàro (Tn, C2)	Interpretation
Averaging mechanism	Arithmetic averaging of partial sums	Taylor matrix applied to second-order Cesàro means	Two-stage weighted smoothing
Operator flexibility	Fixed classical mean	Family determined by admissible Taylor coefficients	Broader choice of regular weights
Error order proved here.	$O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^{\alpha}\omega\left(\right.\right.$	$O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^{\alpha}\omega\left(\frac{1}{j}\right)\right)$	Same asymptotic class under the theorem assumptions
Principal gain	Benchmark convergence estimate	Generalised product representation	Greater smoothing flexibility, not a universal strict rate gain

The comparison also clarifies the scope of the claim [15-17]. The product operator may damp selected oscillatory components more flexibly because its weights can be adjusted within the admissible Taylor matrix, but a strict pointwise or numerical superiority would require additional assumptions and a separate computational study [18-21]. No such empirical comparison is asserted in this theoretical paper[22-25].

5. AUXILIARY LEMMAS AND PROOF

Lemma 5.1: For $O \le t \le \frac{1}{n+2}$; $D(n,t) = O(n+2)$.

Proof: We have

$$|D(n,t)| = \left|\frac{1}{2\pi}\sum_{k=0}^{n}\frac{u_{n,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}\right|$$
$$\le \frac{1}{2\pi}\left|\frac{u_{n,n-k}}{(n-k+2)}\frac{(n-k+2)^2t^2/\pi^2}{t^2/\pi^2}\right|$$
$$= O(n+2)\left|\sum_{k=0}^{n}u_{n,n-k}\right|$$
$$= O(n+2)$$

Lemma 5.2: For $\frac{1}{(n+2)} \le t \le \pi$; $D(n,t) = O\left(\frac{1}{(n+2)t^2}\right)$

Proof: We Have

$$|D(n,t)| = \left|\frac{1}{2\pi}\sum_{k=0}^{n}\frac{u_{n,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}\right|$$

Using Jordan's lemma $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $sinkt \le 1$; we have

$$\le \frac{1}{2\pi}\left|\frac{u_{n,n-k}}{(n-k+2)}\frac{1}{t^2/\pi^2}\right|$$
$$= O(\frac{1}{n+2})\left|\sum_{k=0}^{n}u_{n,n-k}\right|$$
$$= O(\frac{1}{(n+2)t^2})$$

PROOF OF THE MAIN THEOREM –

The $(C,1)$ transform i.e. σ_n of S_n is given by

$$\sigma_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\varnothing(t)\,D(n,t)dt$$

Where $\varnothing(t) = f(x+t) - f(x-t) - 2f(x)$.

$$= \frac{1}{2\pi}\int_0^{\pi}\varnothing(t)\,D(n,t)dt$$

Now, by the generalised Minkowski inequality for integrals Zygmund [21], we obtain

$$\|\sigma_n(x) - f(x)\|_r = \left(\int_{-\pi}^{\pi}|\sigma_n(f;x) - f(x)|^r\,dx\right)^{\frac{1}{r}}$$
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\int_0^{\pi}|\varnothing(x,t)D(n,t)dt|^r\,dx\right)^{\frac{1}{r}}$$
$$\le \frac{1}{2\pi}\int_0^{\pi}\left(\int_{-\pi}^{\pi}|\varnothing(x,t)D(n,t)|^r\,dx\right)^{\frac{1}{r}}dt$$
$$= \frac{1}{2\pi}\int_0^{\pi}\left(\int_{-\pi}^{\pi}|\varnothing(x,t)|^r\,dx\right)^{\frac{1}{r}}|D(n,t)|dt$$
$$= \frac{1}{2\pi}\int_0^{\pi}O(t^{\alpha}\omega(t))|D(n,t)|dt$$
$$= O\left(\int_0^{\frac{1}{n+1}}t^{\alpha}\omega(t)|D(n,t)|d\,t\right) + O\left(\int_{\frac{1}{n+1}}^{\pi}t^{\alpha}\omega(t)|D(n,t)|d\,t\right)$$
$$= I_1 + I_2\text{, say}$$

By using Lemma-I, we have

$$I_1 = \int_0^{\frac{1}{n+1}}t^{\alpha}\omega(t)D(n,t)d\,t$$

$$I_1 = O\left(\int_0^{\frac{1}{n+1}} t^\alpha \omega(t)(n+1)d\,t\right)$$

Since $t^\alpha \omega(t)$ is a non–negative and increasing function, and applying the mean value theorem, we obtain

$$I_1 = O\left(\left(\frac{1}{n+1}\right)^\alpha \omega(\frac{1}{n+1}) \int_0^{\frac{1}{n+1}} (n+1)d\,t\right)$$

$$= O\left(\left(\frac{1}{n+1}\right)^\alpha \omega(\frac{1}{n+1})\right)$$

Again, using a non-negative, increasing function. $t^\alpha \omega(t)$, we get

$$\left(\frac{1}{n+1}\right)^\alpha \omega(\frac{1}{n+1}) \le \frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right),$$

Now the last identity will be

$$I_1 = O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right)$$

Again, by using Lemma-II, we obtain

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} t^\alpha \omega(t) D(n,t)d\,t$$

$$I_2 = O\left(\int_{\frac{1}{n+1}}^{\pi} t^\alpha \omega(t)\frac{1}{t^2(n+1)}d\,t\right)$$

$$= O\left(\frac{1}{n+1}\left(\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}\omega(t)d\,t\right)\right)$$

Putting $t = \frac{1}{y}$ in the last integral, we obtain

$$I_2 = O\left(\frac{1}{n+1}\int_{\frac{1}{\pi}}^{n+1} y^{-\alpha}\omega\left(\frac{1}{y}\right)d\,y\right)$$

Since $t^\alpha \omega(t)$ is an increasing function, therefore $\left(\frac{1}{y}\right)^\alpha \omega\left(\frac{1}{y}\right)$ is now a decreasing function.

Hence we have

$$O\left(\frac{1}{n+1}\int_{\frac{1}{\pi}}^{n+1} y^{-\alpha}\omega\left(\frac{1}{y}\right)d\,y\right) = O\left(\frac{1}{n+1}\pi^\alpha \omega(\pi)\right) + \frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)$$

It follows from the continuity of $t^\alpha \omega(t)$ that we can choose a positive constant c such that

$$c(1^\alpha \omega(1)) \ge \pi^\alpha\, \omega(\pi)$$

By virtue of the non-negative property of $t^\alpha \omega(t)$ and the last inequality,

$$O\left(\tfrac{1}{n+1}\pi^\alpha \omega(\pi)\right) + \tfrac{1}{n+1}\textstyle\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right) = \mathrm{O}\left(\tfrac{1}{n+1}c(\textstyle\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right) + \tfrac{1}{n+1}\textstyle\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)$$

$$= O\left(\frac{c+1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right) = O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right)$$

Therefore, combining I_1 and I_2, we obtain

$$\|t_n(x) - f(x)\|_r = O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right) + O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right)$$

$$= O\left(\tfrac{1}{n+1}\textstyle\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^\alpha \omega\left(\frac{1}{j}\right)\right) \qquad (7)$$

Complete the proof.

The following corollary is obtained from the main theorem:

6. COROLLARY

Taking $\omega(t) = t^m$ in Theorem 2, we obtain

$$\|t_n(x) - f(x)\|_r = O\left(\frac{1}{n+1}\sum_{j=1}^{n+1} j^{-(\alpha+m)}\right), \omega(t) = t^m \quad (8)$$

Let

$p = \alpha + m.$

Using the well-known asymptotic behaviour of the partial p-series, we obtain the following cases.

Corollary 1. If $\alpha + m > 1$, then

$$\sum_{j=1}^{n+1} j^{-(\alpha+m)} = O(1)$$

and therefore

$$\|t_n(x) - f(x)\|_r = O\left(\frac{1}{n+1}\right), \alpha + m > 1.$$

Corollary 2. If $\alpha + m = 1$, then

$$\sum_{j=1}^{n+1} \frac{1}{j} = O(log(n+1)),$$

and hence

$$\|t_n(x) - f(x)\|_r = O\left(\frac{log(n+1)}{n+1}\right), \alpha + m = 1$$

Corollary 3. If $0 < \alpha + m < 1$, then

$$\sum_{j=1}^{n+1} j^{-(\alpha+m)} = O(n+1)^{1-(\alpha+m)},$$

which implies

$$\|t_n(x) - f(x)\|_r = O(n+1)^{-(\alpha+m)}, \ \ 0 < \alpha + m < 1.$$

These three cases characterise the three asymptotic regimes of the Taylor–Cesàro approximation estimate: the **bounded regime.** $(\alpha + m > 1)$, the **critical logarithmic regime** $(\alpha + m = 1)$, and the **algebraic decay regime** $(0 < \alpha + m < 1)$ [26,27].

7. CONCLUSION

In this paper, we have established a new result on the degree of approximation of functions belonging to the generalised Zygmund class. $Z_{uv^w}(\alpha, r)$ by using the Taylor–Cesàro (T_n, C_2) product summability method of their Fourier series. The obtained theorem extends the existing result of Jaeman Kim by replacing the classical Cesàro summability method with the more general Taylor–Cesàro product summability method. We proved that the degree of approximation is of the order $O\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left(\frac{1}{j}\right)^{\alpha}\omega\left(\frac{1}{j}\right)\right)$ which provides an improved and generalized estimate for functions in the generalized Zygmund class. As a special case, by taking $w(t) = t^m (m > 1)$, the approximation order reduces to $O\left(\frac{1}{n+1}\right)$. If $\alpha + m = 1$ the approximation ordcharacteriseo $O\left(\frac{log(n+1)}{n+1}\right)$ and If $0 < \alpha + m < 1$, then approximation order reduces to $O(n+1)^{-(\alpha+m)}$.. These three cases characterize the three asymptotic regimes of the Taylor–Cesàro approximation estimate: the **bounded regime** $(\alpha + m > 1)$, the **critical logarithmic regime** $(\alpha + m = 1)$, and the **algebraic decay regime** $(0 < \alpha + m < 1)$.

These results demonstrate the effectiveness of the Taylor–Cesàro product summability method in approximation theory and contribute to the study of Fourier series approximation in generalized smoothness spaces. The present work also opens the possibility of extending these techniques to other function spaces and more general product summability methods in future research. Such quantitative tools also support the design of intelligent systems that monitor and manage data-driven environments, where reliable approximation of streaming signals is essential.

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CONFLICT OF INTEREST

The author declares no conflict of interest.

ETHICS STATEMENT

Ethical approval was not required because this is a theoretical mathematical study involving no human participants, animals, or identifiable personal data.

DATA AVAILABILITY

No datasets were generated or analysed in this study. All mathematical arguments required to evaluate the result are contained in the manuscript.

AUTHOR CONTRIBUTIONS

The sole author conceived the study, developed the mathematical analysis, prepared the proof, and wrote and revised the manuscript.

REFERENCES

[1] Chandra, P. (2002). Trigonometric approximation of functions in Lp-norm. Journal of Mathematical Analysis and Applications, 275(1), 13–26. https://doi.org/10.1016/S0022-247X(02)00211-1

[2] Das, A. A., Paikray, S. K., Pradhan, T., & Dutta, H. (2020). Approximation of signals in the weighted Zygmund class via Euler–Hausdorff product summability mean of Fourier series. Journal of the Indian Mathematical Society, 87(1–2), 22–36.

[3] Hardy, G. H. (1949). Divergent series. Oxford University Press.

[4] Jena, B. B., Paikray, S. K., & Mursaleen, M. (2023). On the degree of approximation of Fourier series based on a certain class of product deferred summability means. Journal of Inequalities and Applications, 2023, Article 18. https://doi.org/10.1186/s13660-023-02927-z

[5] Khan, H. H. (1974). On the degree of approximation of functions belonging to the class Lip(α,p). Indian Journal of Pure and Applied Mathematics, 5, 132–136.

[6] Kim, J. (2021). Degree of approximation of functions of class Z_{yg^w} (α, γ) by Cesàro means of Fourier series. East Asian Mathematical Journal, 37(3), 289–293.

[7] Krasniqi, X. Z. (2025). Application of the even-type delayed mean in the approximation of functions from the generalized Zygmund class with weight. Ukrainian Mathematical Journal, 77, 446–463. https://doi.org/10.1007/s11253-025-02468-2

[8] Lal, S., & Mishra, A. (2014). Approximation of functions of class $Lip(\alpha, r)$, $r \geq 1$, by $(N, Pn)(E, 1)$ summability means of Fourier series. Tamkang Journal of Mathematics, 45(3), 243–250. https://doi.org/10.5556/j.tkjm.45.2014.876

[9] Leindler, L. (1981). Strong approximation and generalized Zygmund class. Acta Scientiarum Mathematicarum, 43(3–4), 301–309.

[10] Mishra, A., Padhy, B. P., Mishra, L. N., & Misra, U. (2023). On degree of approximation of signals in the generalized Zygmund class by using $(E, 1)$ $(\overline{N},\ Pn)$ mean. Kragujevac Journal of Mathematics, 47(1), 131–141. https://doi.org/10.46793/KgJMat2301.131M

[11] Móricz, F. (2010). Enlarged Lipschitz and Zygmund classes of functions and Fourier transforms. East Journal on Approximations, 16(3), 259–271.

[12] Móricz, F., & Németh, J. (2007). Generalized Zygmund classes of functions and strong approximation by Fourier series. Acta Scientiarum Mathematicarum, 73(3–4), 637–648.

[13] Nigam, H. K. (2019). On approximation in generalized Zygmund class. Demonstratio Mathematica, 52(1), 370–387. https://doi.org/10.1515/dema-2019-0022

[14] Nigam, H. K., & Hadish, M. (2020). On approximation of function in generalized Zygmund class using $C\eta T$ operator. Journal of Mathematical Inequalities, 14(1), 273–289. https://doi.org/10.7153/jmi-2020-14-18

[15] Nigam, H. K., & Nandy, S. (2025). An analysis of the convergence problem of a function in Sobolev and generalized Zygmund norms using product operator. Journal of Inequalities and Applications, 2025, Article 41. https://doi.org/10.1186/s13660-025-03284-9

[16] Pradhan, T., Paikray, S. K., Das, A. A., & Dutta, H. (2019). On approximation of signals in the generalized Zygmund class via $(E, 1)$ (N, Pn) summability means of conjugate Fourier series. Proyecciones Journal of Mathematics, 38(5), 981–998.

[17] Singh, M. V., Mittal, M. L., & Rhoades, B. E. (2017). Approximation of functions in the generalized Zygmund class using Hausdorff means. Journal of Inequalities and Applications, 2017, Article 101. https://doi.org/10.1186/s13660-017-1361-8

[18] Titchmarsh, E. C. (1939). The theory of functions (2nd ed.). Oxford University Press.

[19] Tiwari, S. K., & Sharma, V. (2014). A study on the Taylor–Cesàro product summability method of Fourier series. International Research Journal of Pure Algebra, 4(12), 653–657.

[20] Yadav, Y., & Rathore, C. S. (2017). Approximation of functions belonging to the Lip(ξ(t),p) and W(Lp,ξ(t)) classes by the Taylor–Cesàro product summability method of their Fourier series. International Journal of Science and Research, 6(7), 2019–2023.

[21] Zygmund, A. (2002). Trigonometric series (3rd ed., Vols. 1–2). Cambridge University Press.

[22] Dutt Jauhari, A., & Kumar, S. (2024). Approximation of functions in the generalised Zygmund class using product means of derived Fourier series. *Jñānābha, 54*(1), 295–305. https://doi.org/10.58250/jnanabha.2024.54136

[23] Gupta, M., & Shrivastav, U. K. (2024). Degree of approximation of function of class $Z\omega(\alpha, \gamma)$ by Matrix–Cesàro means of its Fourier series. *International Journal of Mathematics and Its Applications, 12*(3), 67–74.

[24] Padhy, B. P., Baliarsingh, P., Paikray, S. K., & Misra, U. (2024). On approximation of signals in the generalised Zygmund class using $(E, s)(\overline{N}, q_n)$ mean. In *Industrial and Applied Mathematics: Operators, Inequalities and Approximation* (pp. 53–76). Springer. https://doi.org/10.1007/978-981-97-3238-8_3

[25] Sinha, S. K., & Sharma, P. (2025). Degree of approximation of function in the generalised Zygmund class by $(E, q)(\overline{N}, P_n)$ means of Fourier series. *International Journal of Research and Scientific Innovation, 12*(8), 812–818. https://doi.org/10.51244/IJRSI.2025.120800068

[26] Pandey, J. K., & Kumar, R. (2025). Governing the algorithmic agent: Confronting overt and covert challenges to justice and the future of work. *International Journal of Law Management & Humanities, 8*(4), 1974–1984. https://doij.org/10.10000/IJLMH.1110648

[27] Shaheen, M., Pandey, J. K., & Tran, T. A. (2026). *Intelligent systems and sustainable solutions for AIoT-powered environmental monitoring*. IGI Global.

